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1980 J. Phys. A: Math. Gen. 13 141

(http://iopscience.iop.org/0305-4470/13/1/015)

J. Phys. A: Math. Gen., 13 (1980) 141–148. Printed in Great Britain

Non-linear coupling of quantum theory and classical gravity

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Received 26 February 1979

Abstract. We discuss the possibility that the non-linear evolution proposed earlier for a relativistic quantum field theory may be related to its coupling to a classical gravitational field. Formally, in the Schrödinger picture, we show how both the Schrödinger equation and Einstein's equations (with the expectation value of the energy-momentum tensor on the right) can be derived from a variational principle. This yields a non-linear quantum evolution. Other terms can be added to the action integral to incorporate explicit non-linearities of the type discussed previously. We discuss briefly the possibility of giving a meaning to the resulting equation in a Heisenberg or interaction-like picture.

1. Introduction

In a previous publication (Kibble 1978) one of us has proposed a non-linear generalisation of quantum mechanics, and exhibited a class of relativistic models of this type. It was suggested that the quantum non-linearity might well be related in some way to gravity and its associated non-linearity of space–time. To explore this idea further we discuss here the coupling of a quantum field theory (with or without the non-linear generalisation) to a classical gravitational field.

On the question of quantising the gravitational field, the point of view we adopt is that, once the possibility of a non-linear quantum evolution equation is admitted, there is no necessity to quantise gravity. We treat the gravitational field purely classically. It is described by a metric tensor $g_{\mu\nu}$ obeying Einstein's equations with a suitably defined expectation value of the quantum energy-momentum tensor on the right-hand side.

One of the chief difficulties in extending the previous discussion to a curved space–time is that it was expressed in the language of the Schrödinger picture, which is not really appropriate and makes the preservation of covariance difficult. We shall therefore proceed initially in a rather formal way, considering 'unitary' transformations between one picture and another that do not, in fact, exist.

We begin in § 2 by considering in a given curved space–time an ordinary quantum field theory without quantum non-linearities (but possibly with interaction), formulated initially in the Heisenberg picture. We formally transform to the Schrödinger picture, and show (in § 3) that on this level both the Schrödinger equation and Einstein's equations, with the expectation value of $T_{\mu\nu}$ on the right-hand side, can be derived from a common variational principle. This derivation has the virtue of guaranteeing the validity of the consistency conditions such as the Bianchi identities.

0305-4470/80/010141+08\$01.00

As Mielnik (1974) has emphasised, this coupling to a classical gravitational field introduces an intrinsic non-linearity into the quantum theory. Because quantum superpositions are no longer preserved in time it becomes possible in principle to measure 'non-quadratic' observables, those whose probability functions are not expressible as quadratic functions of normalised state vectors. As he puts it, 'either the gravitation is not classical or quantum mechanics is not orthodox'. Here we assume the latter.

In addition to this inevitable non-linearity we can, if we wish, introduce further explicit nonlinearities by adding suitable terms to the action integral. This is discussed in § 4. It is possible that such terms may have a role to play in making the theory better behaved.

To try to give the resulting equations a meaning we then (in § 5) transform back to something akin to an interaction picture, in which the operators carry all the time dependence of the usual interacting field theory while the states vary in time only because of explicit non-linearities. If no such extra terms are added, we recover precisely the Heisenberg picture. There remain severe problems in the way of a rigorous formulation. These are discussed briefly.

2. Field theory in the Schrödinger picture

We consider a four-dimensional globally hyperbolic manifold M of signature $(-+++)$, whose metric $g_{\mu\nu}$ for the moment we take to be given *a priori*. A global slicing into space-like surfaces is then always possible. We choose one such, a family $\sigma(t)$ given locally by equations of the form

$$x^\mu = x^\mu(\xi^1, \xi^2, \xi^3, t)$$

where ξ^r are intrinsic coordinates, such that the normal is everywhere time-like, i.e. there exists a vector field n^μ satisfying

$$n_\mu n^\mu = -1 \qquad n_\mu x^\mu_{,r} = 0$$

where $x^\mu_{,r} = \partial x^\mu/\partial \xi^r$. Denoting the derivative with respect to t at fixed ξ^r by a dot, the lapse function N and shift vector N^r are defined by

$$\dot{x}^\mu = Nn^\mu + N^r x^\mu_{,r}.$$

What we seek is a formula which determines the 'state of the system' on $\sigma(t+\mathrm{d}t)$ given the state on $\sigma(t)$.

Consider first a quantum field theory without non-linear generalisation. For simplicity, we consider only a single scalar field $\phi(x)$. The discussion could readily be generalised to cover other cases. Its dynamics is described by a Lagrangian density function $\mathscr{L}(\phi, \partial_\mu\phi, g_{\mu\nu})$, for example

$$\mathscr{L} = |g|^{1/2}(-\tfrac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \tfrac{1}{2}m^2\phi^2 - \tfrac{1}{24}h\phi^4) \tag{1}$$

where $g = \det g_{\mu\nu}$ and h is the coupling constant. In the Heisenberg picture the field equations for ϕ are given by variation of the operator action integral

$$W_\phi = \int \mathscr{L}\, \mathrm{d}^4x. \tag{2}$$

The corresponding energy-momentum tensor $T^{\mu\nu}$ may be defined by

$$-\tfrac{1}{2}|g|^{1/2}T^{\mu\nu}(x)=\frac{\delta W_\phi}{\delta g_{\mu\nu}(x)}=\frac{\partial \mathscr{L}(x)}{\partial g_{\mu\nu}(x)} \tag{3}$$

although of course this is ill-defined until a regularisation scheme has been adopted. For the particular choice (1) this yields

$$|g|^{1/2}T_{\mu\nu}=-|g|^{1/2}\partial_\mu\phi\,\partial_\nu\phi-g_{\mu\nu}\mathscr{L}.$$

Formally, we may pass from this manifestly covariant formalism to a Schrödinger picture associated with the specific slicing $\sigma(t)$ of space–time by applying a unitary transformation. We suppose that the two pictures coincide on the surface $\sigma(t_0)$. Then the corresponding unitary operator $U(t, t_0)$ should satisfy the equation

$$\mathrm{i}\frac{\partial}{\partial t}U(t,t_0)=H(t)U(t,t_0) \tag{4}$$

where the surface-dependent Hamiltonian is given by

$$H(t)=-\int_{\sigma(t)}\mathrm{d}\sigma_\mu T^\mu_\nu\dot{x}^\nu. \tag{5}$$

Here $\mathrm{d}\sigma_\mu=n_\mu\gamma^{1/2}\,\mathrm{d}^3\xi$, with $\gamma=\det\gamma_{rs}$, where γ_{rs} is the metric induced on $\sigma(t)$,

$$\gamma_{rs}=g_{\mu\nu}x^\mu_{,r}x^\nu_{,s}.$$

It must be emphasised that (4) has at best a formal significance. In general there is no unitary operator satisfying this equation; and indeed $H(t)$ may not exist. Nonetheless the derivation has heuristic value, and we shall pursue it further. Later we return to the question of how to give the resulting formalism a precise mathematical meaning.

The canonical conjugate to ϕ is defined as usual by

$$\pi=\partial\mathscr{L}/\partial\dot{\phi}. \tag{6}$$

Explicitly, for the model described by (1), it is

$$\pi=N^{-1}\gamma^{1/2}(\dot{\phi}-N^r\phi_{,r}).$$

The field equations in Hamiltonian form, including the relation (6) between $\dot{\phi}$ and π, can be derived from variation of the action integral

$$W_{\phi\pi}=\int\mathrm{d}t\Big(\int\mathrm{d}^3\xi\,\pi\dot{\phi}-H\Big). \tag{7}$$

Of course, $W_{\phi\pi}$ reduces to W_ϕ when π is chosen to satisfy

$$\delta W_{\phi\pi}/\delta\pi=0 \tag{8}$$

whose solution is (6). It follows that

$$\frac{\delta W_\phi}{\delta g_{\mu\nu}(x)}=\frac{\delta W_{\phi\pi}}{\delta g_{\mu\nu}(x)}$$

where the functional derivative on the right is at constant ϕ and π, but evaluated for values satisfying (8). Since the first term in (7) is independent of $g_{\mu\nu}$ we therefore obtain, using (3),

$$\frac{\delta}{\delta g_{\mu\nu}(x)}\int \mathrm{d}t\, H(t)=\tfrac{1}{2}|g|^{1/2}T^{\mu\nu}(x). \tag{9}$$

This relation may be verified by explicit computation. It will be useful later.

If we make the formal unitary transformation defined by the solution $U(t, t_0)$ of (4), we obtain a transformed state vector $|\psi(t)\rangle = U(t, t_0)|\psi_0\rangle$ which satisfies Schrödinger's equation

$$\mathrm{i}|\dot{\psi}(t)\rangle = H(t)|\psi(t)\rangle. \tag{10}$$

Correspondingly, the transformed operators $U(t, t_0)\phi(x)U^{-1}(t, t_0)$ and $U(t, t_0)\pi(x)U^{-1}(t, t_0)$ become independent of t, functions of the intrinsic coordinates ξ^r only.

Note that the transformation to the Schrödinger picture depends not only on the slicing of space–time but also on the parametrisation of the slices. A time-dependent change of intrinsic coordinates leads to a different Schrödinger picture.

The equation (10) can also be written in a local form which shows how the state changes under small displacements of the space-like hypersurface $\sigma(t)$. Writing

$$|\dot{\psi}\rangle = \int_{\sigma(t)} \mathrm{d}^3\xi\, \dot{x}^{\mu}\frac{\delta}{\delta x^{\mu}}|\psi\rangle \tag{11}$$

we may express (10) as

$$\mathrm{i}\frac{\delta}{\delta x^{\mu}}|\psi\rangle = -\gamma^{1/2}n_{\mu}T^{\mu}{}_{\nu}(x)|\psi\rangle. \tag{12}$$

3. The action integral

The equation (10) can be derived in various ways from a variational principle. In a previous publication (Kibble 1979) one of us showed how quantum dynamics could be expressed as a Hamiltonian flow on the space Σ of instantaneous pure states, which is essentially a projective Hilbert space, the set of rays in (a dense subspace of) the Hilbert space $\mathscr{H}$. However the corresponding symplectic two-form ω, though closed, is not exact. There exists no analogue of the classical canonical one-form $p_i\, \mathrm{d}q^i$, and hence no possibility of a Lagrangian formalism. One can, of course, derive the Schrödinger equation from a Lagrangian using not Σ but $\mathscr{H}$. However, in that form, one cannot introduce any non-linearity, such as coupling to a classical gravitational field, without violating the invariance of the equations of motion under the transformation $|\psi\rangle \to \lambda|\psi\rangle$ of the state vector.

One can escape this dilemma by using neither Σ nor $\mathscr{H}$, but rather the unit sphere $\mathscr{H}_1$ in $\mathscr{H}$. On $\mathscr{H}_1$ a suitable canonical one-form may be defined, namely

$$\theta = \mathrm{Im}\,\langle \mathrm{d}\psi|\psi\rangle.$$

However, it does not induce a symplectic structure on $\mathscr{H}_1$ because $\omega = -\mathrm{d}\theta$ is necessarily degenerate in the sense that it vanishes on the vector field which generates the phase transformation $|\psi\rangle \to \mathrm{e}^{\mathrm{i}\alpha}|\psi\rangle$.

The restriction to $\mathscr{H}_1$ can be imposed by using a Lagrange multiplier. Thus we take the action integral to be

$$W_\psi = \int \mathrm{d}t\,[\mathrm{Im}\langle\dot\psi|\psi\rangle - \langle\psi|H|\psi\rangle + \alpha(\langle\psi|\psi\rangle - 1)]. \tag{13}$$

Variation of the Lagrange multiplier α yields the constraint

$$\langle\psi|\psi\rangle = 1 \tag{14}$$

while variation of $|\psi\rangle$ yields the Schrödinger equation

$$\mathrm{i}|\dot\psi(t)\rangle = H(t)|\psi(t)\rangle - \alpha(t)|\psi(t)\rangle. \tag{15}$$

The degeneracy is exhibited in the indeterminancy of the Lagrange multiplier α. Physically, of course, (15) is equivalent to (10), because an overall phase in the state vector is unobservable. In practice, it may be convenient to remove the arbitrariness by a suitable convention. This amounts to a choice of the zero point of H.

Next, we wish to show that if a purely gravitational action integral

$$W_\mathrm{g} = \frac{1}{16\pi k}\int \mathrm{d}^4x\,|g|^{1/2}R \tag{16}$$

is added to (13), Einstein's equations result. Of course,

$$\frac{\delta W_\mathrm{g}}{\delta g_{\mu\nu}} = -\frac{1}{16\pi k}|g|^{1/2}G^{\mu\nu}$$

where $G^{\mu\nu}$ is the Einstein tensor. Moreover the expectation value of (9) is

$$\frac{\delta W_\psi}{\delta g_{\mu\nu}} = -\tfrac{1}{2}|g|^{1/2}\langle\psi|T^{\mu\nu}|\psi\rangle. \tag{17}$$

Thus from the variational principle,

$$\delta(W_\mathrm{g} + W_\psi) = 0 \tag{18}$$

we obtain Einstein's equations,

$$G^{\mu\nu} = -8\pi k\langle\psi|T^{\mu\nu}|\psi\rangle \tag{19}$$

as well as the Schrödinger equation (15) and the normalisation condition (14).

Bianchi's identities require

$$\langle\psi|T^{\mu\nu}|\psi\rangle_{;\nu} = 0 \tag{20}$$

which are an expression of the general coordinate invariance of the theory. One of the chief virtues of the derivation from an action principle is that these conditions are guaranteed. They are non-trivial because the states are, of course, time-dependent. However, they can also be verified by direct computation (Kramer 1976).

4. Incorporation of explicit non-linearities

Because the classical gravitational field obtained by solving (19) reacts back on the quantum state, via (15), the quantum evolution is intrinsically nonlinear.

In addition, if we wish, we can incorporate extra explicit nonlinearities by adding terms to the action integral. Such terms, if suitably chosen, may serve to make the theory better behaved. For example, we could add a term like

$$-\tfrac{1}{8}\lambda \int d^4x\, |g|^{1/2} \langle\psi(t)|\phi^2(x)|\psi(t)\rangle^2 \tag{21}$$

to generate nonlinearities of the type considered previously (Kibble 1978).

Here, however, we are more interested in nonlinearities that involve gravity in an essential way. Specifically, we require that the nonlinearity should disappear from the Schrödinger equation when the space–time is flat. One way to achieve that is by including in (21) a factor of the scalar curvature R. More generally, we may take our extra contribution to the action integral to be of the form

$$\int d^4x\, |g|^{1/2} R F(\langle f(\phi)\rangle_\psi) \tag{22}$$

where F and f are suitably chosen functions, and

$$\langle f(\phi)\rangle_\psi = \langle\psi(t)|f[\phi(x)]|\psi(t)\rangle.$$

Note that higher powers of R in (22) can be excluded by the requirement that the field equations be no more than second-order. For the same reason we cannot introduce terms involving, say, $R^{\mu\nu}\langle\partial_\mu\phi\,\partial_\nu\phi\rangle$.

Adding the term (22) to the action integral changes both the Schrödinger equation (15) and Einstein's equations (19). The former becomes

$$i|\dot\psi(t)\rangle = H_\psi(t)|\psi(t)\rangle - \alpha(t)|\psi(t)\rangle \tag{23}$$

with

$$H_\psi(t) = H(t) + H_\psi^{nl}(t)$$

and

$$H_\psi^{nl}(t) = -\int d^3\xi\, N\gamma^{1/2} R F'(\langle f(\phi)\rangle_\psi) f(\phi) \tag{24}$$

where the prime denotes a derivative. Correspondingly, the Einstein equations (19) become

$$-\frac{1}{16\pi k} G_{\mu\nu} = \tfrac{1}{2}\langle T_{\mu\nu}\rangle_\psi + G_{\mu\nu}F - F_{;\mu;\nu} + g_{\mu\nu}F^{;\lambda}_{;\lambda} \tag{25}$$

Notice that the effect of one term on the right-hand side of (25) can be thought of as a state-dependent change in the gravitational constant. The equation can be written in the alternative form

$$G_{\mu\nu} = -\frac{8\pi k}{1+16\pi k F}[\langle T_{\mu\nu}\rangle_\psi - 2F_{;\mu;\nu} + 2g_{\mu\nu}F^{;\lambda}_{;\lambda}]. \tag{26}$$

This is somewhat reminiscent of Brans–Dicke theory (and also of asymptotic freedom). As in the Brans–Dicke case it would be possible to recover a theory with fixed gravitational constant by making a suitable scale transformation (Dicke 1962). However, in our case, the compensating terms involving derivatives of F would be extremely complicated.

Because the term (22) added to the action integral is the integral of a scalar density, the conditions (20) are preserved. Now, of course, it is the right-hand side of (25)—or equivalently of (26)—which has vanishing covariant divergence.

Specific solutions of the equations (23) and (26) will be examined in a future paper.

5. Interpretation

It is almost certain that the Schrödinger equation (23) cannot be given a rigorous mathematical meaning as it stands. In general, a covariant approach seems more likely to succeed than one based on the Schrödinger picture.

In the theory without explicit nonlinearities, we can return to the Heisenberg picture, in which the field operator $\phi(x)$ obeys the field equations

$$-|g|^{-1/2}\partial_\mu(|g|^{1/2}g^{\mu\nu}\partial_\nu\phi)+m^2\phi+\tfrac{1}{6}h\phi^3=0. \tag{27}$$

Einstein's equations retain the form (19), but of course the states are now time-independent.

When non-linearities of the type discussed in § 4 are included, it seems very hard to recover a Heisenberg picture. What we can do, however, is to recover something like an interaction picture. More precisely, we may reverse the transformation effected by $U(t, t_0)$ discussed in § 2. This throws all the time dependence related to the ordinary linear terms in the Schrödinger equation back onto the operators, so that they obey the equation (27). At the same time the state vector has a time dependence due solely to the explicit non-linearity, namely

$$\mathrm{i}|\dot{\psi}(t)\rangle = H_\psi^{nl}(t)|\psi(t)\rangle - \alpha(t)|\psi(t)\rangle. \tag{28}$$

It is also possible to write this equation in a local form like (12), namely

$$\mathrm{i}\frac{\delta}{\delta x^\mu}|\psi\rangle = n_\mu\gamma^{1/2}RF'(\langle f(\phi)\rangle_\psi)f(\phi)|\psi\rangle. \tag{29}$$

However, because the right-hand side is proportional to n_μ (corresponding to a diagonal contribution to the energy-momentum tensor), it can also be written in a form like that of the Tomonoga equation, namely

$$\mathrm{i}\frac{\delta}{\delta\sigma(x)}|\psi\rangle = RF'(\langle f(\phi)\rangle_\psi)f(\phi)|\psi\rangle. \tag{30}$$

Several problems need to be solved to give these equations a meaning. First, in H_ψ^{nl}, and on the right-hand side of Einstein's equations (19) or (25), there are expectation values of quadratic (or higher) functions of the field operators. To give these a precise significance we have to adopt some kind of regularisation scheme. Second, in solving the field equations (27) we of course encounter the usual ultraviolet divergences so that we need to solve the highly non-trivial problem of renormalisation in the presence of a gravitational background. Finally, it is far from certain that the equation (28) for the remaining time dependence of the state vectors would have solutions in the strict sense. It may be better to think of $\psi(t)$ not as a state vector in a Hilbert-space representation but rather as a state on the algebra of field operators: if it is possible to consider the observables at a fixed time as forming an algebra, then $\psi(t)$ may properly be thought of as a linear functional on that algebra.

Acknowledgments

We wish to thank Dr C J Isham and Mr S Rouhani for helpful comments on an earlier draft of this paper. One of us (TWBK) is indebted to the Japan Society for the Promotion of Science, and to Professor S Kamefuchi for hospitality at the University of Tsukuba where part of this work was performed.

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